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LAPLACE TYPE INTEGRALS: TRANSFORMATION TO STANDARD FORM AND UNIFORM ASYMPTOTIC EXPANSIONS*

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Abstract. Integrals are considered which can be transformed into the Laplace integral

$$F_\lambda(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} f(t)\, dt,$$

where f is holomorphic, z is a large parameter, $\mu = \lambda/z$ is a uniformity parameter, $\mu \geqslant 0$. A uniform asymptotic expansion is given with error bounds for the remainders. Applications are given for special functions, with a detailed analysis for a ratio of gamma functions. Further applications are mentioned for Bessel functions and parabolic cylinder functions. Analogue results are given for loop integrals in the complex plane.

1. Introduction. We consider Laplace integrals of the form

$$F_\lambda(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} f(t)\, dt \tag{1.1}$$

where f is holomorphic in a domain Ω that contains the non-negative reals in its interior; λ and z are real or complex variables for which $F_\lambda(z)$ is properly defined. We are interested in the asymptotic expansion of $F_\lambda(z)$ for $z \to \infty$, which is uniformly valid with respect to the parameter $\mu := \lambda/z$. This is earlier considered in Temme (1983).

The present paper gives results for integrals with the same asymptotic phenomena and for which a non-trivial transformation is required to bring these integrals into the standard form (1.1). After this transformation the function f of (1.1) usually depends on μ. In view of this aspect we generalize the previous paper.

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This generalization is understood and motivated by considering the following integrals:

$$\begin{aligned}
&\int_0^\infty e^{-zx}[x(x+1)]^{\lambda-1}\,dx, && \text{Modified Bessel function,}\\
&\int_0^\infty e^{-zx}[x(x+1)^\nu]^{\lambda-1}\,dx, && \text{Whittaker function,}\\
&\int_0^\infty e^{-z[x+ax^2]}x^{\lambda-1}\,dx, && \text{Parabolic cylinder function,}\\
&\int_0^\infty e^{-zx}[1-e^{-x}]^{\lambda-1}\,dx, && \text{Beta function.}
\end{aligned} \tag{1.2}$$

The nature of the asymptotic expansion of these integrals for $z \to \infty$ with $\mu = \lambda/z$ as a uniformity parameter in $[0, \infty)$ is the same as that of (1.1). However, due to the transformation of these integrals into (1.1), the theory of our earlier paper needs modifications.

To describe the method for (1.1) we use positive λ and z. The function f is expanded at $t = \mu$, at which point $t^\lambda e^{-zt}$ is maximal. We write

$$f(t) = \sum_{s=0}^{\infty} a_s(\mu)(t-\mu)^s \tag{1.3}$$

which is substituted in (1.1) to give the expansion

$$F_\lambda(z) \sim z^{-\lambda} \sum_{s=0}^{\infty} a_s(\mu) P_s(\lambda) z^{-s}, \qquad z \to \infty, \tag{1.4}$$

where $P_s(\lambda)$ are polynomials. The first few are $P_0(\lambda) = 1$, $P_1(\lambda) = 0$, $P_2(\lambda) = \lambda$, $P_3(\lambda) = 2\lambda$, $P_4(\lambda) = 3\lambda(\lambda + 2)$. In our earlier paper we discussed the asymptotic nature of (1.4) and we constructed error bounds. The error bounds in the present paper are new and may be more realistic.

The integrals in (1.2) can be written in the form

$$\int_0^a q(x)^{\lambda-1} e^{-zp(x)} h(x)\, dx \tag{1.5}$$

and in fact this type of integral is the starting point of the present investigations. Remark that for the examples of (1.2) p and q are positive increasing functions. Hence one of them can be taken as a new variable of integration.

In Sec. 2 we transform (1.5) into (1.1), with f depending on μ. The necessary modifications of earlier results is given in Sec. 3. The remaining sections contain applications for the special functions in (1.2), with in Sec. 4 a detailed analysis for the Beta integral. These results are also important for a future paper on the incomplete Beta function.

The applications considered here do not give essentially new expansions for the Beta function, parabolic cylinder function and modified Bessel function. For the last case the expansion is, in some sense, equivalent to that given by Olver (1974, Ch. 10) with as starting point the differential equation for the Bessel function. There is a different role for the parameters, however. In Olver's approach the Bessel function $K_\nu(\nu z)$ is considered for $\nu \to \infty$, and uniformity with respect to z. Here we write $K_{\mu z}(z)$ with $z \to \infty$ and μ as a uniformity parameter. Both expansions can be transformed into each other.

Olver developed fundamental methods for obtaining rigorous and realistic error bounds for uniform asymptotic expansions. In almost all cases the starting point is a differential equation. It is important to develop a theory for integrals. Interesting results in this field are obtained and reviewed by Wong (1980).

Terminology. We call a variable *fixed* when it is independent of z and μ. The argument or phase of a complex number z is denoted by $ph\ z$.

2. Transformation to standard form. We consider the transformation of (1.5) into (1.1). We suppose temporarily that λ and z are positive. As a preparatory step we take $p(x) = x$, since the conditions on p and q in the general case (1.5) make it possible to consider either p or q as a new variable of integration. In fact we consider

$$\frac{1}{\Gamma(\lambda)} \int_0^\infty q(x)^{\lambda-1} e^{-zx} h(x)\, dx. \tag{2.1}$$

2.1. *Main assumptions*. The assumptions are:

(2.2.a) Ω is a connected domain of the complex x-plane with

$$\inf_{\substack{x \geqslant 0 \\ \omega \in \partial\Omega}} |x - \omega| = d,$$

where d is a fixed positive number;

(2.2.b) q and h are holomorphic in Ω, they are not depending on λ and z; $q(0) = 0$, other possible zeros of q are outside Ω; q is real and increasing on $[0, \infty)$; by redefining λ and h we take $q'(0) = 1$;

(2.2.c) (2.1) should converge for sufficiently large z and all $\lambda > 0$;

(2.2.d) the function $x - \mu \ln q(x)$ is convex on $(0, \infty)$ and its unique simple positive saddle-point $x_0(\mu)$, $\mu > 0$, is an increasing function on $(0, \infty)$,

$$\lim_{\mu \to 0^+} x_0(\mu) = 0, \qquad \lim_{\mu \to \infty} x_0(\mu) = \infty;$$

possible other saddle-points are outside Ω.

Remark 2.1. The saddle-point $x_0(\mu)$ is found by solving the equation

$$q(x) = \mu q'(x), \qquad \mu = \lambda/z. \tag{2.3}$$

In (2.2.d) we assume that the logarithmic derivative of q is a decreasing function on $(0, \infty)$ with limiting values $+\infty$ (at 0^+) and 0 (at $+\infty$).

2.2. *The transformation*. The integral (2.1) is transformed in the standard form (1.1) by the mapping $x \to t(x)$ defined by

$$x - \mu \ln q(x) = t - \mu \ln t + A(\mu), \tag{2.4}$$

where $A(\mu)$ is a function to be determined. We observe that the right-hand side has a saddle-point at $t = \mu$ and that

$$\frac{dx}{dt} = \frac{q(x)(t-\mu)}{t[q(x) - \mu q'(x)]}. \tag{2.5}$$

The prime in $q'(x)$ denotes a derivative with respect to x.

We see that dx/dt is finite and non-zero for each $x \in [0, \infty)$, except possibly when $x = 0$, $x = x_0$ or $t = 0$, $t = \mu$. Therefore, we require for the mapping $x \to t(x)$ the correspondences

$$x = 0 \leftrightarrow t = 0, \quad x = x_0(\mu) \leftrightarrow t = \mu, \quad x = +\infty \leftrightarrow t = +\infty, \tag{2.6}$$

with the expectation that with these relations dx/dt will be finite and non-zero at $t = 0$, $t = \mu$ also. $A(\mu)$ is determined by substituting $t = \mu$ in (2.4), from which we obtain

$$A(\mu) = x_0 - \mu \ln q(x_0) - \mu + \mu \ln \mu. \tag{2.7}$$

The analytical aspects of the transformations (2.4) are discussed below. First we consider the result, which reads

$$\frac{e^{-zA(\mu)}}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} f(t)\, dt, \tag{2.8}$$

$$f(t) = \frac{h(x)t}{q(x)} \frac{dx}{dt} = h(x) \frac{t-\mu}{q(x) - \mu q'(x)}. \tag{2.9}$$

2.3. *The regularity of the transformation.* In Sec. 3 it is assumed that f is holomorphic as a function of t in a μ-dependent domain Ω_t of the complex t-plane and that it is a regular function of μ, $\mu \geqslant 0$. In this section we establish that this is the case when $q(x)$ and $h(x)$ satisfy the conditions given in 2.1.

We shall need to know the behaviour of x_0 as $\mu \to 0$. The conditions on q allow us to write

$$q(x) = x + q_2 x^2 + q_3 x^3 + \cdots \tag{2.10}$$

and this expansion has a fixed positive radius of convergence. From the implicit function theorem for analytic functions it follows that the solution of (2.3) is an analytic function of μ and that for small $|\mu|$ the series

$$x_0(\mu) = \mu + x_2 \mu^2 + x_3 \mu^3 + \cdots \tag{2.11}$$

has a positive radius of convergence. The first coefficients are $x_2 = q_2$, $x_3 = 2q_3$.

The function $A(\mu)$ defined in (2.7) has the expansion

$$A(\mu) = -q_2 \mu^2 - q_3 \mu^3 + \cdots. \tag{2.12}$$

It is easily shown that A is analytic at $\mu = 0$ and hence the series has a positive radius of convergence.

A complication in the proof of the regularity of the mapping (2.4) is that the saddle-points of the functions in (2.4) at $x = x_0$, $t = \mu$ tend to 0 when $\mu \to 0$. In the limit $\mu = 0$ both saddle-points vanish and the mapping reduces to the identity. Moreover, the (removable) logarithmic singularities of the functions in (2.4) disappear in the limit $\mu = 0$. These phenomena make the mapping for small values of μ and x (or t) quite complicated.

To prove the regularity of the mapping (2.4) we introduce a function $\tau(x)$ by writing

$$t = \left[\frac{\mu}{x_0} + (x - x_0)\tau\right] x. \tag{2.13}$$

This matches the points ($x = 0$, $t = 0$) and also ($x = x_0$, $t = \mu$). Moreover, it gives the proper linear relation between x and t for small μ and x. Note that μ/x_0 is finite and regular at $\mu = 0$ (see (2.11)).

Substituting (2.13) in (2.4), we try to solve for τ. We show that τ can be expanded in powers of x (when $|x|$ is small) with coefficients regular in μ. The substitution yields

$$x(1 - \mu/x_0) - \mu\ln[q(x)/x] = x\tau(x - x_0) - \mu\ln(\mu/x_0) + A(\mu)$$
$$-\mu\ln[1 + x_0\tau(x - x_0)/\mu].$$

By expanding the last log-term we can write this as

$$\frac{\phi(x)}{(x - x_0)^2} - \tau - \frac{x_0^2\tau^2}{\mu}\sum_{s=2}^{\infty}\frac{(-1)^s}{s}[x_0\tau(x - x_0)/\mu]^{s-2} = 0, \tag{2.14}$$

with

$$\phi(x) = x(1 - \mu/x_0) - \mu\ln\frac{q(x)}{x} - A(\mu) + \mu\ln(\mu/x_0).$$

From (2.3) and (2.7) it follows that $\phi(x_0) = \phi'(x_0) = 0$. Hence $\phi(x)/(x - x_0)^2$ is analytic at $x = x_0$, uniformly in μ (μ small). Note that $\mu/x_0 \sim 1$, $\phi(0) = O(\mu^2)$, $\mu \to 0$. Also, the series in (2.14) represents an analytic function for small values of μ, τ and x. When $\mu = 0$, the mapping (2.4) reduces to $x = t$. Hence τ of (2.13) has to vanish in the limit $\mu = 0$.

Now we are ready to apply the following implicit function theorem (Chow & Hale (1982, p. 36)).

THEOREM 2.1. Consider the equation $F(w, z) = 0$, where F: $\mathbf{C} \times \mathbf{C}^2 \to \mathbf{C}$ is analytic in a neighborhood of $(0,0)$ and $F(0,0) = 0$, $D_w F(0,0) \neq 0$. Then there exists $\varepsilon > 0$ such that for every z, $|z| < \varepsilon$, the equation $F(w, z) = 0$ has a unique solution $w(z)$ which is analytic in a neighborhood of zero.

We take $w = \tau$, $z = (x, \mu)$ and we denote the left-hand side of (2.14) by $F(w, z)$. It follows that τ, and hence t, is analytic in x and μ for small values of these variables, and that the mapping (2.4) is uniformly one-to-one for small values of x and μ. It also follows that x is an analytic function of (t, μ) in a neighborhood of $(0, 0)$. For the remaining values of μ and x the regularity is much easier to prove.

The first term in the expansion

$$x(t) = c_1(\mu)t + c_2(\mu)t^2 + \cdots$$

follows from (2.4) and (2.10), giving

$$c_1(\mu) = \lim_{t\to 0}\frac{x}{t} = \lim_{t\to 0}\exp\left[\frac{x - t - A(\mu)}{\mu}\right] = \exp[-A(\mu)/\mu],$$

which limit is indeed analytic in μ at $\mu = 0$.

Remark that the implicit function theorem can be applied also for $z = (x - x_0, \mu)$, since $\phi''(x_0) = O(\mu)$, $\mu \to 0$. The limiting value of dt/dx at $x = x_0$ follows from an application of l'Hôpital's rule on the right-hand side of (2.5). The result is (by using (2.3))

$$\left.\frac{dt}{dx}\right|_{x=x_0} = +\{1 - \mu^2 q''(x_0)/q(x_0)\}^{1/2} \tag{2.15}$$

where the sign of the square root is $+$, since we assume that t is an increasing function of x. The assumption in (2.2.d) that x_0 is a simple saddle-point implies that this expression does not vanish. For small values of μ, the expansions (2.10), (2.11) give

$$\left.\frac{dt}{dx}\right|_{x=x_0} = 1 - q_2\mu + \cdots$$

where the series has a positive radius of convergence.

The main result of this section is that the function f of (2.8) is analytic in t and μ, where t and μ range in connected domains containing the positive reals in their interior. Here and in the following sections we consider applications with complex analytic functions. However, the regularity of the mapping can be proved also for real functions $h(x)$ and $q(x)$ belonging to continuity classes $C^k[0, \infty)$.

3. Asymptotic expansion. In this section we reconsider the asymptotic expansion of (1.1) as given in Temme (1983); z is the large parameter, $\mu := \lambda/z$ is the uniformity parameter, $\mu \geqslant 0$. The conditions on f are now more general than in the earlier analysis.

3.1. *Assumptions on f.* We suppose that f is holomorphic in a connected μ-dependent domain Ω_t of the complex t-plane, with the condition (2.2.a) with x replaced by t. Let R_μ denote the radius of convergence of (1.3). Then we suppose moreover that

$$R_\mu^{-1} = O([1+\mu]^{-\kappa}), \qquad \mu \geqslant 0, \quad (\kappa \text{ fixed}, \kappa \geqslant 1/2). \tag{3.1}$$

We assume that f has the following growth condition in Ω_t: there is a real fixed number p such that

$$\sup_{t\in\Omega_t} (1+|t|)^{-p}|f(t)| \tag{3.2}$$

is bounded for all finite values of μ, $\mu \geqslant 0$.

Remark 3.1. For $\kappa < 1/2$ the singularities of f are too close to the saddle-point $t = \mu$. This case will be excluded here.

Remark 3.2. The conditions on the location of the singularities of f and (3.2) are quite natural for the examples in (1.2), when transformed into the standard form.

3.2. *Asymptotic scale.* The coefficients $a_s(\mu)$ of (1.3) can be written as

$$a_s(\mu) = \frac{1}{2\pi i}\int_{C_r} \frac{f(t)}{(t-\mu)^{s+1}}\,dt, \tag{3.3}$$

where C_r is a circle with centre μ and radius $r(1+\mu)^\kappa$; r may depend on μ but it should be uniformly bounded from zero and it should be small enough to keep C_r inside Ω_t. Using (3.2) we obtain the bound

$$|a_s(\mu)| \leqslant r^{-s}M_r(\mu)(1+\mu)^{-s\kappa} \tag{3.4}$$

where

$$M_r(\mu) = \left[1+\mu+r(1+\mu)^\kappa\right]^p \sup_{t\in C_r\cup[0,\infty)} (1+|t|)^{-p}|f(t)|. \tag{3.5}$$

For later purposes, $M_r(\mu)$ describes also the growth of $|f(t)|$ on $[0, \infty)$. We introduce the sequence $\{\psi_r\}$ by defining

$$\psi_s = \psi_s(z, \mu) = M_r(\mu)(1+\mu)^{-s\kappa} z^{-s}(1+\lambda)^{s/2}, \qquad s = 0, 1, 2, \ldots. \tag{3.6}$$

THEOREM 3.1. $\{\psi_s\}$ is an asymptotic sequence as $z \to \infty$, uniformly in $\mu \geqslant 0$.

Proof. $\psi_{s+1}/\psi_s = (1+\lambda)^{1/2}(1+\mu)^{-\kappa} z^{-1} \leqslant z^{-1/2}$ when $\mu \geqslant 0$ and $z \geqslant 1$. □

Remark 3.3. The value of κ is important here, the value 1/2 being critical:

(i) The theorem is not true when $\kappa < 1/2$.

(ii) When $\kappa > 1/2$, $\{\psi_s\}$ is also an asymptotic sequence as $\mu \to \infty$, uniformly with respect to $z \geqslant z_0 > 0$ (z_0 *fixed*).

(iii) When $\kappa = 1/2$, we have $\psi_s = M_r(\mu) z^{s/2} \chi_s$, with $\chi_s = [(1+\mu z)/(z+\mu z)]^{s/2} \leqslant 1$ ($s \geqslant 1$). $\{\chi_s\}$ is not an asymptotic sequence as $\mu \to \infty$.

3.3. *Asymptotic nature of the expansion* (1.4). The expansion (1.4) is written as

$$z^\lambda F_\lambda(z) \sim \sum_{s=0}^{\infty} a_s(\mu) P_s(\lambda) z^{-s}; \qquad \{\psi_s(z, \mu)\} \quad \text{as } z \to \infty, \tag{3.7}$$

where for the notation we refer to Olver (1974, p. 25) or to Erdélyi & Wyman (1963). The functions $P_s(\lambda)$ are polynomials in λ defined as

$$P_s(\lambda) = \frac{1}{\Gamma(\lambda)} \int_0^\infty x^{\lambda-1} e^{-x}(x-\lambda)^s \, dx, \qquad s = 0, 1, 2, \cdots, \tag{3.8}$$

of which the first few are given after (1.4). They follow the recursion $P_{s+1}(\lambda) = s[P_s(\lambda) + \lambda P_{s-1}(\lambda)]$, $s \geqslant 1$. In the proof of the following theorem we also use

$$\tilde{P}_s(\lambda) = \frac{1}{\Gamma(\lambda)} \int_0^\infty x^{\lambda-1} e^{-x} |x-\lambda|^s \, dx, \qquad s \geqslant 0. \tag{3.9}$$

By applying Laplace's method it is found that

$$\tilde{P}_s(\lambda) \sim \pi^{-1/2}(2\lambda)^{s/2} \Gamma\left(\frac{s+1}{2}\right), \qquad \lambda \to \infty. \tag{3.10}$$

To prove (3.7), we need a representation of the remainder. Let us write (1.3) in the form

$$f(t) = \sum_{s=0}^{n-1} a_s(\mu)(t-\mu)^s + (t-\mu)^n R_n(t, \mu). \tag{3.11}$$

Then we obtain for (1.4)

$$F_\lambda(z) = z^{-\lambda}\left[\sum_{s=0}^{n-1} a_s(\mu) P_s(\lambda) z^{-s} + z^{-n} E_n(z, \lambda)\right], \tag{3.12}$$

where the remainder E_n is defined by

$$z^{-n} E_n(z, \lambda) = \frac{z^\lambda}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt}(t-\mu)^n R_n(t, \mu) \, dt. \tag{3.13}$$

THEOREM 3.2. The expansion (3.7) is a uniform asymptotic expansion for $z \to \infty$, the uniformity holding with respect to $\mu \in [0, \infty)$.

Remark 3.4. According to the definition of generalized (uniform) asymptotic expansion, we have to prove

$$z^{-n}E_n(z,\mu) = O[\psi_n(z,\mu)], \qquad n = 0,1,\cdots, \tag{3.14}$$

as $z \to \infty$, uniformly in $\mu \geqslant 0$.

3.4. *Proof of Theorem* 3.2. The interval of integration in (3.13) is split up as follows

$$[0,\infty) = \Delta_- \cup [t_-, t_+) \cup \Delta_+, \tag{3.15}$$

where

$$\Delta_- = [0, t_-], \quad \Delta_+ = [t_+, \infty), \quad t_\pm = \mu \pm r_1(1+\mu)^\kappa, \quad 0 < r_1 < r, \quad r_1 \text{ fixed}, \tag{3.16}$$

with r as in (3.3). When t_- happens to be negative we replace it by 0. For $t \in [t_-, t_+)$ we can write with C_r as in (3.3)

$$R_n(t,\mu) = \frac{1}{2\pi i}\int_{C_r} \frac{f(\tau)}{(\tau - t)(\tau-\mu)^n}\,d\tau. \tag{3.17}$$

For $\tau \in C_r$ we have $|\tau - t| \geqslant (r - r_1)(1+\mu)^\kappa$. Thus we obtain as in (3.4)

$$|R_n(t,\mu)| \leqslant \frac{M_r(\mu)(1+\mu)^{-n\kappa}}{r^{n-1}(r-r_1)}. \tag{3.18}$$

Hence the integral over $[t_-, t_+)$ in (3.13) gives a contribution which is bounded by

$$\begin{aligned} &\frac{z^\lambda}{\Gamma(\lambda)} \frac{M_r(\mu)(1+\mu)^{-n\kappa}}{r^{n-1}(r-r_1)} \int_{t_-}^{t_+} t^{\lambda-1}e^{-zt}|t-\mu|^n\,dt \\ &\quad = M_r(\mu)(1+\mu)^{-n\kappa}z^{-n}\tilde{P}_n(\lambda)O(1), \qquad z \to \infty, \end{aligned} \tag{3.19}$$

uniformly in $\mu \geqslant 0$. Using (3.10), we conclude that

$$z^{-n}E_n(z,\lambda) = I_- + I_+ + O[\psi_n(z,\mu)], \tag{3.20}$$

$z \to \infty$, uniformly in $\mu \geqslant 0$, where $I_\pm$ are the contributions to (3.13) due to $t \in \Delta_\pm$.

For these t-values $R_n(t,\mu)$ is written as

$$(t-\mu)^n R_n(t,\mu) = f(t) - \sum_{s=0}^{n-1} a_s(\mu)(t-\mu)^s.$$

The proof is finished when we have shown that

$$\frac{z^\lambda}{\Gamma(\lambda)}\int_{\Delta_\pm} t^{\lambda-1}e^{-zt}g(t)\,dt = O[\psi_n] \quad \text{as } z \to \infty, \tag{3.21}$$

uniformly in $\mu \geqslant 0$, where $g(t)$ is $|f(t)|$ or $|a_s(\mu)(t-\mu)^s|$ $(0 \leqslant s \leqslant n-1)$. In fact we will prove more than in (3.21), namely that the integrals in (3.21), and hence $I_\pm$, are asymptotically equal to 0 with respect to the scale $\{\psi_s\}$.

To give an estimate for $I_\pm$, it is important to do so for the function

$$\Phi(t) = \frac{1}{\Gamma(\lambda)} z^\lambda t^\lambda e^{-zt}, \qquad t \geqslant 0.$$

For each $\mu \geqslant 0$ it attains its maximum value in Δ_-, Δ_+ at the endpoints t_-, t_+. So we proceed with $\Phi(t_\pm)$.

LEMMA 3.1. We have the following bound

$$\Phi(t_\pm) \leqslant [\lambda/(2\pi)]^{1/2} \exp\left[-\rho_\pm z(1+\mu)^{\alpha_\pm(\kappa)}\right]$$

where $\rho_\pm \geqslant \rho_0 > 0$ (ρ_0 fixed) and $\alpha_+(\kappa) = \min(\kappa, 2\kappa - 1)$, $\alpha_-(\kappa) = 2\kappa - 1$.

Proof. It easily follows that $\Phi(t_+) \leqslant [\lambda/(2\pi)]^{1/2} \exp[-z\phi_+(\mu)]$, where we used $1/\Gamma(\lambda) \leqslant [\lambda/(2\pi)]^{1/2} e^{\lambda}\lambda^{-\lambda}$ and where

$$\phi_+(\mu) = r_1(1+\mu)^\kappa\left[1 - \frac{\ln(1+y)}{y}\right], \qquad y = \frac{r_1(1+\mu)^\kappa}{\mu}.$$

Using

$$1 - \frac{\ln(1+y)}{y} \geqslant \frac{y/2}{1+y} \qquad (y \geqslant 0)$$

we have

$$\phi_+(\mu) \geqslant \frac{1/2 r_1^2(1+\mu)^{2\kappa}}{\mu + r_1(1+\mu)^\kappa}.$$

When $\kappa > 1$, we use $\mu + r_1(1+\mu)^\kappa \leqslant (r_1+1)(\mu+1)^\kappa$, from which we obtain

$$\phi_+(\mu) \geqslant \frac{r_1^2(1+\mu)^\kappa}{2(r_1+1)}.$$

When $1/2 \leqslant \kappa \leqslant 1$, we use $\mu + r_1(1+\mu)^\kappa \leqslant (r_1+1)(1+\mu)$, resulting into

$$\phi_+(\mu) \geqslant \frac{r_1^2(1+\mu)^{2\kappa-1}}{2(r_1+1)}.$$

This proves the lemma for $\Phi(t_+)$.

For $\Phi(t_-)$ we take $\rho_- = +\infty$ for the μ-values that make t_- negative, and we continue with $t_- > 0$. We have $\Phi(t_-) \leqslant [\lambda/(2\pi)]^{1/2} \exp[-z\phi_-(\mu)]$, where

$$\phi_-(\mu) = -\mu[y + \ln(1-y)], \qquad y = \frac{r_1(1+\mu)^\kappa}{\mu},$$

with the condition $t_- > 0$, or $\mu > r_1(1+\mu)^\kappa$, which for y implies $0 \leqslant y < 1$. Expanding $\ln(1-y)$ we obtain

$$\phi_-(\mu) \geqslant 1/2 r_1^2(1+\mu)^{2\kappa}/\mu \geqslant 1/2 r_1^2(1+\mu)^{2\kappa-1}.$$

This proves the lemma. □

The bounds for the integrals in (3.21) are essentially given in the following lemma.

LEMMA 3.2. Consider the following integrals

$$G_\pm(\alpha, q) = \int_{\Delta_\pm} |t-\alpha|^q t^{\lambda-1} e^{-zt}\, dt,$$

where $(\alpha, q) = (\mu, s)$ or $(-1, p)$. Then we have the bounds

$$G_\pm(\alpha, q) \leqslant r_1^{-1}\Phi(t_\pm)|t_\pm - \alpha|^q(1+\mu)^{-\kappa}[z-\sigma]^{-1}, \qquad z > \sigma.$$

Here σ is a fixed real number, which is given below.

Proof. When $t_- = 0$, the Δ_- integral vanishes, together with $\Phi(t_-)$. So we can proceed with $t_- > 0$. We compute a real number σ which satisfies

$$|(t-\alpha)/(t_\pm - \alpha)|^q \leqslant \left[(t/t_\pm)^{-\mu} e^{t - t_\pm}\right]^\sigma, \qquad t \in \Delta_\pm.$$

Taking logarithms we write this as $\phi(t) \leqslant 0$, $t \in \Delta_\pm$, where

$$\phi(t) = q \ln\left|\frac{t-\alpha}{t_\pm - \alpha}\right| + \mu\sigma \ln(t/t_\pm) - \sigma(t - t_\pm).$$

The derivative $\phi'(t)$ is non-negative (non-positive) on $\Delta_-(\Delta_+)$ when $\phi'(t_-) \geqslant 0$ $(\phi'(t_+) \leqslant 0)$. This yields for σ the inequality

$$\sigma \geqslant \frac{qt_-}{|t_\pm - \alpha|(t_\pm - \mu)}.$$

All combinations of $\pm$, $\alpha = \mu$, $\alpha = -1$ show that the right-hand side is a bounded function of μ ($\mu \geqslant 0$, $\kappa \geqslant 1/2$) and we take σ as the supremum of this function (and of the four combinations). With this bound for $|t - \alpha|^q$ we obtain for $G_\pm$

$$G_\pm(\alpha, q) \leqslant |t_\pm - \alpha|^q \Phi(t_\pm) \int_{\Delta_\pm} t^{-1} T(t)^{z-\sigma}\, dt,$$

where $T(t) = (t/t_\pm)^\mu \exp(t_\pm - t)$. Taking $x = -\ln T(t)$ as a new variable of integration, we have

$$G_\pm(\alpha, q) \leqslant |t_\pm - \alpha|^q \Phi(t_\pm) \int_0^\infty e^{-(z-\sigma)x} \frac{dx}{|t-\mu|}.$$

Replacing $|t-\mu|^{-1}$ by the larger quantity $|t_\pm - \mu|^{-1} = r_1^{-1}(1+\mu)^{-\kappa}$ we arrive at the desired results. □

Now we are ready to establish the final result for $I_\pm$ of (3.20).

LEMMA 3.3.

$$I_\pm \sim 0;\ \{\psi_s(z, \mu)\} \quad \text{as } z \to \infty,$$

uniformly with respect to $\mu \geqslant 0$.

Proof. We consider the integrals in (3.21) for the following cases

(i) $g(t) = |f(t)|$. Using (3.5) we obtain the bound

$$M_r(\mu)\left[1 + \mu + r(1+\mu)^\kappa\right]^{-p} G_\pm(-1, p).$$

The results of Lemma 3.1 and Lemma 3.2 show that this is $o[\psi_m]$, $m = 0, 1, \ldots,$ as $z \to \infty$, uniformly in $\mu \geqslant 0$.

(ii) $g(t) = |a_s(\mu)(t-\mu)^s|$. We now have the bound

$$M_r(\mu) r^{-s}(1+\mu)^{-s\kappa} G_\pm(\mu, s), \qquad s = 0, 1, \ldots, n-1.$$

Again it easily follows that this is $o[\psi_m]$.

This proves the lemma and Theorem 3.2. □

Remark 3.5. When $\kappa > 1/2$, $\Phi(t_\pm)$ is exponentially small as $z \to \infty$, uniformly in $\mu \geqslant 0$. When $\kappa = 1/2$, this is not true. Then the factor $(1+\mu)^{-\kappa}$ in the bound for $G_\pm(\alpha, q)$ in Lemma 3.2 is needed to absorb $\lambda^{1/2}$ which occurs in the bound of $\Phi(t_\pm)$ in Lemma 3.1. See also Remark 3.3 for the peculiar case $\kappa = 1/2$.

3.5. *Error bounds*. Using (3.10), (3.19) and the smallness of $I_\pm$ (see (3.20)) it follows from the proof of Theorem 3.2 that (3.13) satisfies

$$|z^{-n}E_n(z,\lambda)| \leqslant m_n\psi_n(z,\mu), \qquad n = 0,1,\cdots, \tag{3.22}$$

where m_n are fixed and approximately equal to $\pi^{-1/2}2^{1/2n}\Gamma(1/2+n/2)r^{-n}$. This gives an error bound for (3.12).

A more rigorous approach is based on computing the maximal value of $R_n(t,\mu)$ in (3.13). When $a_n(\mu) \neq 0$ and $a_{n+m}(\mu) \neq 0$ we define

$$M_n(\mu) = \sup_{t\geqslant 0} \frac{|R_n(t,\mu)|}{|a_n(\mu)| + \theta_m|a_{n+m}(\mu)(t-\mu)^m|}; \tag{3.23}$$

$m = \max(0, -[n-p])$, where $[\cdot]$ is the entier-function, $\theta_m = 1$ $(m \geqslant 1)$, $\theta_0 = 0$. This choice of m is based on (3.2) and it makes the function at the right-hand side of (3.23) a bounded function of t on $[0,\infty)$. With this definition we obtain

$$|E_n(z)| \leqslant M_n(\mu)\left\{|a_n(\mu)|\tilde{P}_n(\lambda) + \theta_m|a_{n+m}(\mu)|\tilde{P}_{n+m}(\lambda)z^{-m}\right\}. \tag{3.24}$$

The value of $M_n(\mu)$ may be determined by t-values far from the point $t = \mu$. In these cases the bound in (3.24) grossly overestimates the actual error. Therefore it is preferable to seek a majorant that concentrates upon values near $t = \mu$. Modifying Olver's method for Laplace integrals (see Olver (1974, p. 89)) we introduce a number σ_n such that

$$|R_n(t,\mu)| \leqslant M|a_n(\mu)|\left[(t/\mu)^{-\mu}e^{t-\mu}\right]^{\sigma_n} \qquad (0 < t < \infty). \tag{3.25}$$

M is an arbitrary factor exceeding unity.

The best value of σ_n is given by

$$\sigma_n = \sup_{t>0} \chi_n(t,\mu) \tag{3.26}$$

where

$$\chi_n(t,\mu) = \frac{\ln|R_n(t,\mu)/(Ma_n(\mu))|}{t-\mu-\mu\ln(t/\mu)}. \tag{3.27}$$

For small $|t-\mu|$ we have

$$\chi_n(t,\mu) \sim \frac{-\ln M + \frac{a_{n+1}}{a_n}(t-\mu) + O(t-\mu)^2}{(t-\mu)^2 + O(t-\mu)^3}. \tag{3.28}$$

and it follows that σ_n is finite. It depends on μ. In place of (3.24) we derive for $z > \sigma_n$

$$\begin{aligned} |E_n(z)| &\leqslant M|a_n(\mu)|(1-\sigma_n/z)^{-n}|\tilde{P}_n(\lambda-\mu\sigma_n)Q_n, \\ Q_n &= (1-\sigma_n/z)^{-\lambda}\left[(\lambda-\mu\sigma_n)/e\right]^{\mu\sigma_n}\Gamma(\lambda-\mu\sigma_n)/\Gamma(\lambda). \end{aligned} \tag{3.29}$$

In applications σ_n is small. When $z - \sigma_n$ and $\lambda = \mu z$ are large the factor Q_n is close to unity, which follows from the Stirling approximation for the gamma functions. In fact we have

$$Q_n = (1 - \sigma_n/z)^{-1/2}\Gamma^*(\mu(z - \sigma_n))/\Gamma^*(\mu z), \tag{3.30}$$

where

$$\Gamma^*(z) = (z/2\pi)^{1/2} e^z z^{-z}\Gamma(z). \tag{3.31}$$

Example 3.1. With $f(t) = 1/(1 + t)$, (1.1) reduces to the exponential integral. In this case $R_n(t, \mu)/a_n(\mu) = (1 + \mu)/(1 + t)$. Hence (3.23) gives $M_n(\mu) = 1 + \mu$. For large μ this factor is unacceptable large for the bound (3.24). The function $\chi_n(t, \mu)$ of (3.27) is positive on $(0, T)$ with $T = (1 + \mu)/M - 1$. Numerical calculations give with $M = 1.1$ the following table for σ_n. Note that χ_n and σ_n do not depend on n, in this example. We also give the values of Q_n of (3.30) when $z = 5$.

TABLE 1. σ_n and Q_n of (3.29), $f(t) = 1/(1 + t)$, $M = 1.1$, $z = 5$.

μ	σ_n	Q_n	μ	σ_n	Q_n
0.5	0.912	1.114	25	0.206	1.022
1.0	1.207	1.155	50	0.108	1.011
2.5	1.078	1.132	100	0.055	1.006
5.0	0.755	1.086	500	0.012	1.002
7.5	0.570	1.063	1000	0.006	1.001
10.0	0.456	1.050	5000	0.002	1.000
15.0	0.325	1.035	10000	0.001	1.000

Remark 3.6. The exponential integral is considered in our previous paper, sections 3.5 and 6.1. The error bounds of the present section are new and are not given there.

Remark 3.7. Error analysis based on (3.17) (by deforming C_r into a contour around the positive τ-axis) gave rather poor numerical results, compared with (3.24) and (3.29).

To evaluate the bounds in (3.24) and (3.29) it is convenient to have expressions for $\tilde{P}_n(\lambda)$ defined in (3.9). Recall that for n even $\tilde{P}_n(\lambda) = P_n(\lambda)$, the polynomial (3.8). In general we have

$$\tilde{P}_n(\lambda) = \lambda^n n!\left[u_n + (-1)^n v_n\right]/\Gamma(\lambda);$$

u_n, v_n both satisfy the recursion $n\lambda w_n = (n - 1)w_{n-1} + w_{n-2}$ with initial values $u_0 = \Gamma(\lambda, \lambda)$, $v_0 = \gamma(\lambda, \lambda)$, $u_1 = -v_1 = e^{-\lambda}\lambda^{\lambda-1}$. It follows that $\tilde{P}_1(\lambda) = 2e^{-\lambda}\lambda^\lambda/\Gamma(\lambda)$,

$$w_3 = \left[w_0 + (\lambda + 1)w_1\right]/(3\lambda^2),$$

$$w_5 = \left[(6 + 5\lambda)w_0 + (6 + 11\lambda + 2\lambda^2)w_1\right]/(30\lambda^4)$$

where w_s is u_s or v_s. Using the above initial values for u_s and v_s, $\tilde{P}_3$ and $\tilde{P}_5$ easily follow. We have

$$\tilde{P}_3(\lambda) = \frac{2\lambda}{\Gamma(\lambda)}\left[2(\lambda + 1)e^{-\lambda}\lambda^{\lambda-1} + 2\Gamma(\lambda, \lambda) - \Gamma(\lambda)\right].$$

For integer values of a, the incomplete gamma function $\Gamma(a, z)$ is an elementary function:

$$\Gamma(n+1, z) = n!e^{-z} \sum_{m=0}^{n} \frac{z^m}{m!}.$$

3.6. *A related expansion.* In Sec. 5 of Temme (1983) a related expansion for (1.1) is obtained by partial integration. For this expansion the sequence $\{f_s(t)\}$ is used, which is defined by $f_0(t) = f(t)$,

$$f_{s+1}(t) = t \frac{d}{dt} \frac{f_s(t) - f_s(\mu)}{t - \mu}, \qquad s = 0, 1, 2, \cdots. \tag{3.32}$$

Then the expansion reads

$$F_\lambda(z) = z^{-\lambda} \left[\sum_{s=0}^{n-1} f_s(\mu) z^{-s} + z^{-n} E_n^*(z, \lambda) \right], \tag{3.33}$$

$$E_n^*(z, \lambda) = \frac{z^\lambda}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} f_n(t)\, dt. \tag{3.34}$$

The coefficients $f_s(\mu)$ can be expressed in terms of $a_s(\mu)$. We have

$$\begin{aligned} &f_0(\mu) = a_0(\mu), \qquad f_1(\mu) = \mu a_2(\mu), \\ &f_2(\mu) = \mu[3\mu a_4(\mu) + 2a_3(\mu)], \\ &f_3(\mu) = \mu[15\mu^2 a_6(\mu) + 20\mu a_5(\mu) + 6a_4(\mu)]. \end{aligned} \tag{3.35}$$

An interesting point is that in (3.33) the parameters z and μ are separated from each other, whereas in (3.7) the parameter $\lambda = \mu z$ explicitly occurs.

From (3.32) it follows that f_s is holomorphic in Ω_t, as is f itself. By induction it follows

$$f_s(\mu) = (1+\mu)^{qs - 2s\kappa} M_r(\mu) O(1), \qquad \mu \geqslant 0, \tag{3.36}$$

cf. (3.5), where $q = \max(\kappa, 1)$. When $\kappa \geqslant 1/2$, $\{f_s(\mu) z^{-s}\}$ is an asymptotic scale for $z \to \infty$ uniformly with respect to $\mu \geqslant 0$. When $\kappa > 1/2$ the same is true for $\mu \to \infty$, uniformly with respect to z, $z \geqslant z_0 > 0$; z_0 fixed.

An error bound easily follows from (3.34). Let Ω_t contain a sector containing the positive real t-axis in its interior. From (3.2) and Temme (1983, section 2) it follows that

$$\sup_{t \geqslant 0} (1+t)^{s-p} |f_s(t)|$$

is bounded for fixed finite values of μ, $\mu \geqslant 0$. For those n for which $f_n(\mu) \neq 0$, we introduce a number σ_n^* such that

$$|f_n(t)| \leqslant M^* |f_n(\mu)| [(t/\mu)^{-\mu} e^{t-\mu}]^{\sigma_n^*} \qquad (0 < t < \infty), \tag{3.37}$$

where M^* is a fixed arbitrary factor exceeding unity (cf. (3.25)). Then $E_n^*(z, \lambda)$ of (3.27) is bounded by

$$|E_n^*(z, \lambda)| \leqslant M^* |f_n(\mu)| Q_n, \tag{3.38}$$

where Q_n is given in (3.29) (replace σ_n by σ_n^*).

The best value of σ_n^* is given by

$$\sigma_n^* = \sup_{t>0} \frac{\ln|f_n(t)/(M^* f_n(\mu))|}{t - \mu - \mu \ln(t/\mu)}. \tag{3.39}$$

Example 3.2. With $f(t) = 1/(1+t)$, we have

$$f_2(t)/f_2(\mu) = \left[(\mu+1)^2 t(\mu t - \mu - 2)\right]/\left[\mu(\mu-2)(t+1)^3\right], \qquad \mu \neq 2, \mu \neq 0.$$

Take $M^* = 2$. For the μ-values of Table 1, $\mu = 2.5$ gives a positive σ_2^*-value. For the remaining μ-values we can take $\sigma_2^* = 0$.

3.7. *Expansions for loop integrals.* In Temme (1983) loop integrals of the form

$$G_\lambda(z) = \frac{\Gamma(\lambda+1)}{2\pi i}\int_{-\infty}^{(0^+)} e^{zt} t^{-\lambda-1} f(t)\, dt \tag{3.40}$$

were considered. The asymptotic expansion is in this case

$$G_\lambda(z) \sim z^\lambda \sum_{s=0}^{\infty} (-1)^s a_s(\mu) P_s(-\lambda) z^{-s}, \qquad z \to \infty, \tag{3.41}$$

with a_s and P_s as in (1.4). This result remains valid when f depends on μ with the assumptions of section 3.1. The domain of holomorphy of $f(t)$ may be different of course. Again, the expansion holds uniformly with respect to $\mu := \lambda/z$ in $[0, \infty)$. The asymptotic scale is as in (3.6), (3.7). Error bounds may be constructed as in section 3.3. A suitable contour is for that purpose

$$L = \left\{\, t = \rho e^{i\theta} | \rho(\phi) = \mu\phi/\sin\phi,\ -\pi < \phi < \pi \right\}, \tag{3.42}$$

the path of steepest descent for the integral in (3.40).

3.8. *Extension to complex parameters.* In Sec. 2.2 we considered the regularity of the transformation $x \to t(x)$ with emphasis on nonnegative μ-values. It is possible to repeat the analysis for complex μ-values; the essential step, Theorem 2.1, is not restricted to real values. It follows that we can assume that f of (1.1) is a holomorphic function of t and μ in a domain $\Omega_t \times \Omega_\mu \subset \mathbf{C}^2$. Both domains Ω_t, Ω_μ satisfy (2.2.a) with x replaced by t, μ, and for some d.

Complex values of $\mu = \lambda/z$ occur when λ and or z are complex. Let $\theta = ph\, z$, $\nu = ph\, \lambda$, $\chi = ph\, \mu$, with $\nu = \theta + \chi$. For the convergence of (1.1) at $t = 0$ we need $-1/2\pi < \nu < 1/2\pi$. The convergence at $t = \infty$ is determined by θ. Suppose we can rotate the path of integration of (1.1), or deform it at ∞, such that the upper limit is at $\infty e^{i\gamma}$, $-\alpha \leqslant \gamma \leqslant \beta$ where α and β are positive numbers. Then the range for θ is $-1/2\pi - \beta < \theta < 1/2\pi + \alpha$. Given θ in this range and fixing $\gamma \in (-\alpha, \beta)$ we can try to deform the path of integration of (1.1) into a contour $\mathbf{P}$ so that it has the following properties:

(i) $0 \in \mathbf{P}$, $\mu \in \mathbf{P}$, $\infty e^{i\gamma} \in \mathbf{P}$;

(ii) $\mathbf{P}$ lies in Ω_t;

(iii) $\mathrm{Re}\, e^{i\theta}[\phi(t) - \phi(\mu)]$ is positive on $\mathbf{P}$, except at $t = \mu$, and is bounded away from zero as $t \to 0$ or ∞ along $\mathbf{P}$; here $\phi(t) = t - \mu \ln t$.

Let $\nu \in (-1/2\pi, 1/2\pi)$ and $D(\nu, \theta) \subset \Omega_\mu$ be the subset of the points μ for which a path $\mathbf{P}$ can be constructed having the above three properties. Then for fixed values of $\mu \in D(\nu, \theta)$ a theorem of Olver (1974, p. 127) can be used to prove that (1.4) is valid for these values of z and μ. A uniform version of this requires extra conditions, for instance on the location of the singularities of f.

4. A ratio of gamma functions. The expansion of this section is related to

$$\Gamma^*(z) \sim \sum_{k=0}^{\infty} c_k z^{-k}, \qquad z \to \infty, |ph\, z| < \pi, \tag{4.1}$$

$c_0 = 1$, $c_1 = 1/12$, $c_2 = 1/288$, $c_3 = -139/51840$; $\Gamma^*(z)$ is defined in (3.31). By dividing this expansion by a similar one with z replaced by $z(1+\mu)$ we obtain (with $\lambda = \mu z$)

$$\Gamma^*(z)/\Gamma^*(z+\lambda) \sim \sum_{k=0}^{\infty} d_k(\mu) z^{-k} \qquad z \to \infty, \tag{4.2}$$

$d_0(\mu) = 1$, $d_1(\mu) = \mu/[12(1+\mu)]$, $d_2(\mu) = \mu^2/[288(1+\mu)^2]$. It is expected that this expansion holds uniformly with respect to $\mu \geqslant 0$. In this section we give an expansion for $\Gamma(z)/\Gamma(z+\lambda)$ which is related to the above expansion. Starting point is the Beta integral and bounds for the remainder follow from the previous section. The formal above method will not give this information. However, (4.2) can be obtained by the methods of Sec. 3.4, with (4.7) as starting point.

Another known expansion is

$$\Gamma(z)/\Gamma(z+\lambda) \sim z^{-\lambda}\left\{1 - \frac{\lambda(\lambda-1)}{2z} + \frac{\lambda(\lambda^2-1)(3\lambda-2)}{24z^2} + \cdots\right\} \tag{4.3}$$

($z \to \infty$) as given in Olver (1974, p. 118) and in Luke (1969, Vol. I), where more expansions of this kind can be found. All these results lack uniformity with respect to unbounded λ-domains; (4.3) can be used when $\lambda << z^{1/2}$.

4.1. *Uniform asymptotic expansion.* A simple transformation in the Beta integral gives

$$F_\lambda(z) = \Gamma(z)/\Gamma(z+\lambda) = \frac{1}{\Gamma(\lambda)}\int_0^\infty (1-e^{-x})^{\lambda-1} e^{-zx}\, dx \tag{4.4}$$

which is of the form (2.1); q is an entire function and the conditions in (2.2) are readily verified. The saddle-point of (2.2.d) is

$$x_0 = \ln(1+\mu), \qquad \mu = \lambda/z, \tag{4.5}$$

and the transformation (2.4) reads

$$\begin{gathered} x - \mu \ln(1-e^{-x}) = t - \mu \ln t + A(\mu), \\ A(\mu) = (1+\mu)\ln(1+\mu) - \mu. \end{gathered} \tag{4.6}$$

The transformed version of (4.4) is

$$F_\lambda(z) = \frac{e^{-zA(\mu)}}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1} e^{-zt} f(t)\, dt, \tag{4.7}$$

$$f(t) = (t-\mu)/[1-(1+\mu)e^{-x}] = \frac{t}{1-e^{-x}}\frac{dx}{dt}.$$

The first few coefficients of (1.3) are given by

$$\begin{gathered} a_0(\mu) = (1+\mu)^{1/2}, \quad a_1(\mu) = [\mu - 1 + a_0(\mu)]/(3\mu), \quad a_2(\mu) = 1/[12a_0(\mu)], \\ a_3(\mu) = [8\mu^3 + 12\mu^2 - 12\mu - 8 + (8 + 8\mu - 15\mu^2)a_0(\mu)]/[540(1+\mu)\mu^3], \\ a_4(\mu) = [\mu a_2(\mu)/12 - 4(1+\mu)a_3(\mu)]/[6\mu(1+\mu)]. \end{gathered} \tag{4.8}$$

A possible starting point for the computations is the differential equation (obtained from the second line in (2.9))

$$t\tau(1+\mu)\frac{df}{dt} = \left[(1+\mu)t+\tau^2\right]f + (\mu-1)\tau f^2 - \mu f^3,$$

with $\tau = t - \mu$. For $\mu = 0$ the function $f(t)$ reduces to $t/(1-e^{-t})$. Hence, the limiting value $a_s(0)$ satisfies $a_s(0) = (-1)^s B_s/s!$, $s = 0, 1, 2, \cdots$, where B_s are the Bernoulli numbers.

The normalized expansion (1.4) reads (cf. (4.2))

$$\Gamma^*(z)/\Gamma^*(z+\lambda) \sim \sum_{s=0}^{\infty} \frac{a_s(\mu)}{a_0(\mu)} P_s(\lambda) z^{-s} \sim \left[1 + \frac{\lambda}{12z(z+\lambda)} + \cdots\right]. \qquad (4.9)$$

Information on error bounds is given in section 4.3.

4.2. *The logarithmic mapping* (4.6). The singular points of the mapping are the points in the x-plane

$$\xi_m = 2\pi i m, \quad x_n = \ln(1+\mu) + 2\pi i n, \qquad m, n \in \mathbf{Z}\backslash\{0\}. \qquad (4.10)$$

At ξ_m the function $\ln(1-e^{-x})$ is singular and corresponding t-values are at infinity in the right half-plane (when $\mu > 0$). For $\mu = 0$ the points ξ_m are regular.

The points x_n are the zeros of dt/dx. Note that $x_0 = \ln(1+\mu)$ is a regular point. Corresponding values $t_n(\mu) := t(x_n)$ are defined by the equation

$$t_n - \mu \ln t_n = \mu - \mu \ln \mu + 2\pi i n, \qquad n \in \mathbf{Z}\backslash\{0\}. \qquad (4.11)$$

We consider solutions with $|ph\, t_n| \leqslant \pi$. For $\mu \to 0^+$ the points $t_n(\mu)$ approach $2\pi i n$. In the limit $\mu = 0$ these points are regular.

Writing $t = \mu s$, $s = \rho e^{i\theta}$, $\mu > 0$, $\rho > 0$, $-\pi < \theta < \pi$, we obtain from (4.11) the set of equations

$$\begin{cases} \rho\cos\theta - 1 - \ln\rho = 0 \\ \rho\sin\theta - \theta = \omega, \ \omega = 2\pi n/\mu. \end{cases} \qquad (4.12)$$

Let $\omega > 0$. When $0 \leqslant \rho \leqslant 1$ and $\theta \geqslant 0$, then $\rho \sin\theta \leqslant \theta$ and hence the second equation cannot be satisfied. It follows that ρ should be larger than unity, to have roots of (4.12) in the half-plane $\mathrm{Im}\, t > 0$. A more detailed analysis shows that the solutions of (4.12) ($\omega > 0$) in the half-plane $\mathrm{Im}\, t < 0$ belong to a different branch of the many-valued function $t_n(\mu)$, which is implicitly defined by (4.11).

To obtain the asymptotic behaviour of t_n for large μ we introduce $s_n = t_n/\mu$. Then for s_n (4.11) reads

$$s_n - 1 - \ln s_n = 2\pi n i/\mu.$$

Hence, for large μ, $s_n \to 1$ and it easily follows

$$t_n(\mu) \sim \mu + (4\pi i n\mu)^{1/2}, \qquad \mu \to \infty, \qquad (4.13)$$

where $i^{1/2} = e^{\pi i/4}$. The other sign of the square root gives an estimate for a solution belonging to the above mentioned different branch.

It follows that the number κ introduced in Sec. 3, see for instance (3.1), equals $1/2$, which is already predicted by the ratios a_2/a_1, a_1/a_0 of the coefficients in (4.8). For r in

(3.4) and (3.17) we take a positive decreasing function $r(\mu)$, with $r(0) < 2\pi$, $r(\infty) < (2\pi)^{1/2}$, (only $t_{\pm 1}$ are relevant for the singularities of f). The set $\{t_1(\mu) | \mu \geqslant 0\}$ in the complex t-plane is drawn in Fig. 4.1. It cuts the imaginary t-axis at $2\pi i$. The set $\{t_{-1}(\mu) | \mu \geqslant 0\}$ is obtained by reflexion, since $t_1(\mu) = \overline{t_{-1}(\mu)}$. The parametric equation of the curve is given in (4.11) with $n = 1$. The curves for $t_n(\mu)$, $n \geqslant 2$, are located above he curve for $t_1(\mu)$.

The domain Ω_t of holomorphy of f can be taken μ-independent by using for $\operatorname{Re} t > 0$ the curves for $t_{\pm 1}(\mu)$ (see Fig. 4.1) as its boundary with arbitrary extension into $\operatorname{Re} t < 0$, where f has no singularities. For the real variable case this domain suffices. A more general μ-dependent domain Ω_t comprises several Riemann sheets, with branch points $t_{\pm n}(\mu)$ defined in (4.11).

The analytical aspects of the conformal mapping $x \to t(x)$ are well understood when we consider the image of the strip in the complex x-plane

$$S = \{ x = u + iv | u \in \mathbf{R}, |v| < 2\pi \}.$$

The boundary points $\xi_{\pm 1} = \pm 2\pi i$ of S (see (4.10)) are mapped into infinity, the boundary points $x_{\pm 1} = \ln(1 + \mu) \pm 2\pi i$ are mapped into $t_{\pm 1}(\mu)$. In Fig. 4.2 we give the image of a finite part of S. A local analysis at C and D shows that in the t-plane the vertical distance between C and D is approximately $\mu\pi$ (when C and D are close to $2\pi i$ in the x-plane). At $t_1(\mu)$, f has an algebraic singularity,

$$f(t) = O\left[(t - t_1(\mu))^{-1/2} \right], \qquad t \to t_1(\mu).$$

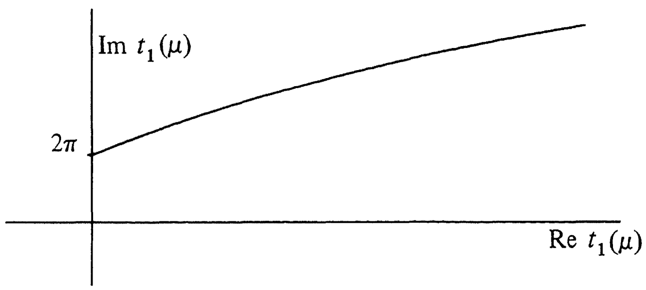

Fig. 4.1. Singular point $t_1(\mu)$, $\mu \geqslant 0$

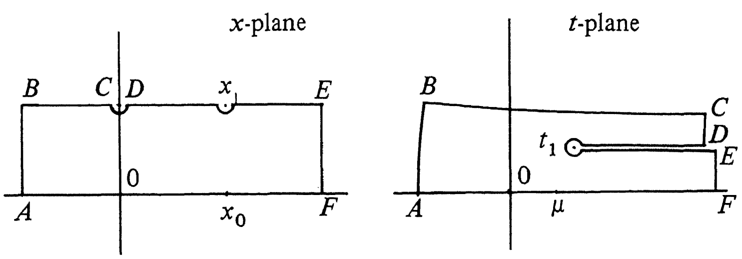

Fig. 4.2. The logarithmic mapping (4.6)

4.3. *Error bounds*. We apply the method of section 3.5 leading to (3.24). It appeared that in this case the bounds $M_n(\mu)$ are slowly varying functions of μ; we checked the cases $n = 2, 3$. In (3.23) $m = 0$ for these values, since $p = 1$ (cf. (3.2) and (4.7)). Numerical values of $M_n(\mu)$ are given in Table 4.1. We also give the ratio

$$\delta_n = \bar{E}_n/E_n, \qquad n = 2, 3,$$

where E_n is the error defined in (3.12) and $\bar{E}_n$ is the a priori computed bound in the right-hand side of (3.24). Since $M_n(\mu)$ is rather close to unity, an approach as leading to (3.29) is not considered here.

TABLE 4.1. Bounds M_n and ratios δ_n; $z = 10$.

μ	$M_2(\mu)$	δ_2	$M_3(\mu)$	δ_3
1	1.015	1.014	1.325	9.48
5	1.081	1.078	1.129	12.8
10	1.125	1.121	1.040	14.2

Observe that δ_3 is much larger then δ_2. An explanation is found in the occurrence of $\tilde{P}_n(\lambda)$ in (3.24). For even n we have $\tilde{P}_n(\lambda) = P_n(\lambda)$. For odd n, $\tilde{P}_n(\lambda)$ may be much larger than $P_n(\lambda)$, and this may overestimate the bound E_n considerably.

Application of the method leading to (3.38) with $n = 1$, $z = 10$ gives Table 4.2; σ_1^* is defined in (3.39) and $\delta_1^* = \bar{E}_1^*/E_1^*$, where E_1^* is the exact error in (3.33) and $\bar{E}_1^*$ the bound in (3.38).

TABLE 4.2. Parameters σ_1^* of (3.39) and ratios δ_1^*; $z = 10$.

		$M^* = 1.1$		$M^* = 1.5$	
μ	E_1^*	σ_1^*	δ_1^*	σ_1^*	δ_1^*
1	0.0042	5.375	1.250	0.999	1.581
5	0.0070	1.697	1.204	0.269	1.515
10	0.0076	1.240	1.172	0.170	1.508

Observe that in this example the ratios δ_1^* are rather close to M^*. Larger z-values will make Q_n of (3.38) closer to unity and hence they will make M^* and δ_1^* more equal.

The case $n = 1$ in this example gives an error bound for the expansion (4.2), taking only the term $d_0(\mu) = 1$.

4.4. *Representation as a loop integral*. The expansion (4.1) has the interesting property that the reciprocal function has the expansion with coefficients $(-1)^k c_k$. This gives for $\Gamma^*(z+\lambda)/\Gamma^*(z)$ the expansion (4.2) with coefficients $(-1)^k d_k(\mu)$. A similar operation is also possible for (4.4) by using

$$G_\lambda(z) := \frac{\Gamma(z+\lambda)}{\Gamma(\lambda)} = \frac{\Gamma(1+\lambda)}{2\pi i(1+\mu)} \int_{-\infty}^{(0^+)} e^{zx}(1-e^{-x})^{-\lambda-1}\,dx, \tag{4.14}$$

$\lambda = \mu/z$. This representation is of the form (3.40) and it can be obtained from Luke (1969, Vol. I, p. 34). Using the transformation (4.6) we obtain

$$G_\lambda(z) = \frac{\Gamma(1+\lambda)e^{zA(\mu)}}{(1+\mu)2\pi i}\int_{-\infty}^{(0^+)} e^{zt}t^{-\lambda-1}f(t)\,dt, \tag{4.15}$$

with $f(t)$ given by (4.7). A suitable contour for integration is given by (3.42). Observe that L is the path of steepest ascent for (1.1) and that $(0, \infty)$ is the path of steepest ascent for (4.15). This makes the expansions of (1.1) and (4.15) quite symmetric. We have (cf. (4.9) and (3.41))

$$\Gamma^*(\lambda + z)/\Gamma^*(z) \sim \sum_{s=0}^{\infty}(-1)^s\frac{a_s(\mu)}{a_0(\mu)}P_s(-\lambda)z^{-s}, \tag{4.16}$$

where the first $a_s(\mu)$ are given in (4.8). This expansion is valid for $z \to \infty$, uniformly with respect to $\mu = \lambda/z$, $\mu \in [0, \infty)$.

4.5. *Extension to complex parameters.* We expect that the expansions (4.9) and (4.16) are valid for complex μ and z values in the range $ph z \in (-\pi, \pi)$, $ph(1+\mu) \in (-\pi, \pi)$. For $\mu = -1$ the mapping (4.6) is not defined; x_0 becomes $-\infty$ in that event. For all remaining complex μ-values the mapping and the coefficients $a_s(\mu)$ are defined properly. A more detailed analysis is needed to trace the singularities of $f(t)$ and the path of integration for complex values of μ. These technical aspects are not considered in this paper.

5. Modified Bessel function $K_\nu(z)$. We give a brief description how the methods of Secs. 2 and 3 can be used to derive an expansion for $K_\nu(z)$ as $z \to \infty$, which is uniformly valid in ν. The starting point is

$$K_\nu(z) = \frac{\pi^{1/2}(z/2)^\nu e^{-z}}{\Gamma(\nu+1/2)}\int_0^\infty e^{-zx}[x(x+2)]^{\nu-1/2}\,dx. \tag{5.1}$$

The saddle-points of $\exp(-zx)[x(x+2)]^\lambda$ are

$$x_0 = e^\gamma - 1, \qquad x_1 = -1 - e^{-\gamma} \tag{5.2}$$

where $\sinh\gamma = \mu$, $\mu = \lambda/z$, $\lambda = \nu + 1/2$; we suppose $z > 0$, $\lambda > 0$. The mapping (2.4) is

$$\begin{aligned} x - \mu\ln[x(x+2)] &= t - \mu\ln t + A \\ A &= \cosh\gamma - (\gamma + \ln 2)\sinh\gamma - 1. \end{aligned} \tag{5.3}$$

With

$$f(t) = (t-\mu)/[x^2 + 2x(1-\mu) - 2\mu] = \frac{t}{x(x+2)}\frac{dx}{dt}, \tag{5.4}$$

we obtain for (5.1) the standard form (1.1) with

$$F_\lambda(z) = \pi^{-1/2}(z/2)^{-\nu}e^{z(A+1)}K_\nu(z). \tag{5.5}$$

The first few coefficients of (1.3) are found to be

$$\begin{aligned} a_0(\mu) &= [2(1+\exp(2\gamma))]^{-1/2}, \\ a_1(\mu) &= \frac{2a_0(\mu)}{3\sinh 2\gamma}\left[\cosh\gamma - 2e^{2\gamma}a_0(\mu)\right]. \end{aligned} \tag{5.6}$$

The function $t(x)$, which is defined by (5.3), is singular at the negative saddle-point x_1 and at $x = -2$. The latter is mapped into ∞ and it has no influence upon the asymptotic expansion. The point x_1 is more interesting. The corresponding point t_1 in the t-plane, giving a singularity for $f(t)$, satisfies the equation

$$t_1 - \mu \ln t_1 = \mu[1 + 2(\gamma - \coth\gamma) - \ln\mu \pm i\pi] \tag{5.7}$$

where $ph x_1 = \pm\pi$; the two signs give conjugate pairs of solutions. For $\mu = 0$ we have $t_1 = x_1 = -2$ (the mapping (5.3) reduces to the identity and t_1 is no longer a singular point). By writing $t_1 = \mu s_1$, (5.7) reads when $\mu \neq 0$:

$$s_1 - \ln s_1 - 1 = 2(\gamma - \coth\gamma) \pm i\pi.$$

Hence, for $\gamma \to \infty$ we have $s_1 = (2\gamma \pm i\pi)(1 + o(1))$, $t_1 \sim \mu[2\ln(2\mu) \pm i\pi]$, $\mu \to +\infty$. It follows that κ of (3.1), (3.4) and (3.6) satisfies $\kappa = 1$.

The expansion for $K_\nu(z)$ reads

$$K_\nu(z) \sim (2\pi/z)^{1/2} e^{-z(\cosh\gamma - \gamma\sinh\gamma)} \sum_{s=0}^{\infty} a_s(\mu) P_s(\lambda) z^{-s}, \tag{5.8}$$

$z \to \infty$, uniformly with respect to μ, $\mu \geqslant 0$. Since κ is greater than $1/2$ it is also an expansion for $\nu \to \infty$, uniformly in $z \geqslant z_0 > 0$ (see Remark 3.3). Inspection of coefficients $a_s(\mu)$ shows that it is allowed to take $z_0 = 0$. The above expansion is related to the expansion given by Olver (1974, Ch. 10.7).

Loop integrals are also available; for instance for the I-function we have

$$\pi^{1/2}(z/2)^\nu e^{-z} I_\nu(z) = \frac{\Gamma(\lambda+1)}{2\pi i} \int_{-\infty}^{(0^+)} e^{zx}[x(x+2)]^{-\lambda-1}\,dx,$$

$= \lambda + 1/2$, which can be transformed by using (5.3) into the standard form (3.40).

6. Parabolic cylinder functions. The methods of Secs. 2 and 3 can be applied to the parabolic cylinder function with integral representation

$$D_{-\nu}(z) = \frac{z^\nu e^{-1/4z^2}}{\Gamma(\nu)} \int_0^\infty e^{-z^2(x+1/2x^2)} x^{\nu-1}\,dx \tag{6.1}$$

$z > 0$, $\nu > 0$. With $\mu = \nu/z^2$, the saddle-points are $x_0 = \sinh^2\gamma$, $x_1 = -\cosh^2\gamma$, where $\mu = 1/4\sinh^2 2\gamma$. The expansion obtained in this way is not essentially new compared with known expansions for parabolic cylinder functions. A corresponding loop integral is also available; it represents the above function for negative values of its argument. The representation is

$$D_{-\nu-1}(-z) = \frac{z^{-\nu} e^{1/4z^2}}{i\sqrt{2\pi}} \int_{-i\infty}^{+i\infty} e^{z^2(x+1/2x^2)} x^{-\nu-1}\,dx, \tag{6.2}$$

where the path cuts the real axis at a positive x-value.

For both functions (6.1) and (6.2) we can obtain related expansions valid for $z \to \infty$ holding uniformly in $\mu = \nu/z^2$, $\mu \geqslant 0$.

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